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Technical Notes

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Compressibility Effect in Vortex Identification

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Introduction

FLOW problems of aircraft, missile dynamics, high-speed transportation, compressible slender- and bluff-body wakes, wingtip vortices, aerodynamics of rotors, propellers and blades, environmental flows, and local meteorology are some examples of aerodynamic problems frequently dominated by vortical structures. Understanding of vortices and vortex dynamics in such flows is of primary interest. Flow modeling is steadily forced to improve the analysis of the given aerodynamic problems and the accuracy of compressible-flow prediction. A considerable number of vortex-identification methods, vortex definitions, and vortex-core visualization techniques have been proposed during the last 25 years [1–21]. The recent study of Kolář [21] has pointed out a variety of general requirements for vortex identification, just one of which is the validity of vortex-identification schemes for compressible flows. In this respect, the region-type definitions of a vortex should be distinguished from the line-type definitions of a vortex core [16]. It should be noted, however, that these methods may be generally combined [22]. It is shown subsequently that from the most popular region-type vortex-identification schemes (Q, Δ, λ_2, and λ_{ci}), only the Δ criterion and the closely associated λ_{ci} criterion are directly extendable to compressible flows.

Compressibility Effect and Widely Used Identification Methods

In this section, the most popular vortex-identification schemes (Q, Δ, λ_2, and λ_{ci}) are examined from the viewpoint of their applicability or extendability to compressible flows (for the λ_2 criterion, it has been already analyzed by Cucitore et al. [12], as briefly mentioned at the end of this section).

Q Criterion

Hunt et al. [3] identified vortices of an incompressible flow as connected fluid regions with a positive second invariant of the velocity-gradient tensor $\nabla\mathbf{u}$ ($\nabla\mathbf{u} = \mathbf{S} + \mathbf{\Omega}$), where $\mathbf{S}$ is the strain-rate tensor, and $\mathbf{\Omega}$ is the vorticity tensor (in tensor notation below the subscript comma denotes differentiation),

$$Q \equiv \tfrac{1}{2}(u_{i,i}^2 - u_{i,j}u_{j,i}) = -\tfrac{1}{2}u_{i,j}u_{j,i} = \tfrac{1}{2}(\|\mathbf{\Omega}\|^2 - \|\mathbf{S}\|^2) > 0 \qquad (1)$$

that is, as the regions in which the vorticity magnitude prevails over the strain-rate magnitude. The norm (or absolute tensor value) $\|\mathbf{G}\|$ of any tensor $\mathbf{G}$ is defined by $\|\mathbf{G}\| = [\mathrm{tr}(\mathbf{G}\mathbf{G}^T)]^{1/2}$. In addition, the pressure in the vortex region is required to be lower than the ambient pressure [however, the pressure condition is, in most cases, covered by condition (1)].

For compressible flows, the Q criterion suffers from ambiguity, as it offers two ways of extension that have clearly different physical meanings: namely, the second invariant of $\nabla\mathbf{u}$ and the quantity $(\|\mathbf{\Omega}\|^2 - \|\mathbf{S}\|^2)/2$. Apart from this ambiguity of the Q criterion, let us first examine the role of compressibility in the quantity $(\|\mathbf{\Omega}\|^2 - \|\mathbf{S}\|^2)/2$, taken as a possible candidate for vortex identification. As the absolute tensor value remains unchanged under an orthogonal transformation, we may, without loss of generality, express $\mathbf{S}$ in the system of principal axes:

$$\mathbf{S} = \begin{pmatrix} a+k & 0 & 0 \\ 0 & b+k & 0 \\ 0 & 0 & c+k \end{pmatrix} \qquad (2)$$

where a, b, and c are the eigenvalues of the strain-rate deviator tensor (i.e., it holds $a + b + c = 0$), k denotes the uniform dilatation, $k > 0$ stands for expansion, and $k < 0$ for compression. It follows that

$$\|\mathbf{S}\|^2 = (a+k)^2 + (b+k)^2 + (c+k)^2 = a^2 + b^2 + c^2 + 3k^2 \qquad (3)$$

Both the contribution of expansion and compression to $\|\mathbf{S}\|^2$ are given by the same positive value $3k^2$ regardless of the sign of k and, consequently, $(\|\mathbf{\Omega}\|^2 - \|\mathbf{S}\|^2)/2$ cannot distinguish between expansion and compression. This conclusion is not physically acceptable: only the opposite effect of expansion and compression on the vortex-identification criteria is natural.

Let us consider the other alternative extension of the original Q criterion to compressible flows given by the second invariant of $\nabla\mathbf{u}$ ($\nabla\mathbf{u}$ is depicted in the system of principal axes of $\mathbf{S}$, and the expression for Q explicitly shows only the terms based on the leading-diagonal elements of $\nabla\mathbf{u}$ depending on k):

$$\begin{aligned} Q &\equiv \tfrac{1}{2}(u_{i,i}^2 - u_{i,j}u_{j,i}) = \tfrac{1}{2}[(a+b+c+3k)^2 - ((a+k)^2 \\ &\quad + (b+k)^2 + (c+k)^2) - \cdots] \\ &= \tfrac{1}{2}[9k^2 - (a^2 + b^2 + c^2 + 3k^2) - \cdots] \end{aligned} \qquad (4)$$

It is apparent that Q cannot distinguish between expansion and compression. Hence, as in the previous case of the alternative quantity $(\|\mathbf{\Omega}\|^2 - \|\mathbf{S}\|^2)/2$, for compressible flows, Q as the second invariant of $\nabla\mathbf{u}$ is not physically acceptable as the vortex-identification criterion. The Q criterion as a whole is not extendable (for two reasons, if including its ambiguity) to compressible flows, provided that it is not a priori redefined in terms of a *deviatoric* part of $\mathbf{S}$ to read $(\|\mathbf{\Omega}\|^2 - \|\mathbf{S}_D\|^2)/2$.

Δ Criterion and λ_{ci} Criterion

Dallmann [1], Vollmers et al. [2], and Chong et al. [4] defined vortices as the regions in which the eigenvalues of $\nabla\mathbf{u}$ are complex

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and the streamline pattern is spiraling or closed in a local reference frame moving with the point. For incompressible flows, the characteristic equation for the eigenvalues λ of $\nabla\mathbf{u}$ reads

$$\lambda^3 + Q\lambda - R = 0 \tag{5}$$

where Q and R are the second and third invariants of $\nabla\mathbf{u}$, Q is given by Eq. (1), $R \equiv \det(u_{i,j})$. To guarantee complex eigenvalues of $\nabla\mathbf{u}$, the discriminant Δ of Eq. (5) should be positive:

$$\Delta = \left(\frac{Q}{3}\right)^3 + \left(\frac{R}{2}\right)^2 > 0 \tag{6}$$

The vortex-identification criterion (6), valid for incompressible flows only, has frequently been explicitly stated in the literature (Jeong and Hussain [8], Kida and Miura [10], Cucitore et al. [12], Chakraborty et al. [18], Haller [19], Zhang and Choudhury [20], and Kolář [21]).

In the case of compressible flows, it is necessary to consider the characteristic equation for the eigenvalues λ of $\nabla\mathbf{u}$ in full; that is,

$$\lambda^3 - P\lambda^2 + Q\lambda - R = 0 \tag{7}$$

where $P \equiv u_{i,i}$. To qualify the examined point as a vortex, in addition to one real eigenvalue, a pair of complex-conjugate eigenvalues should be the solution of Eq. (7). This requirement in terms of the positivity of the discriminant Δ of Eq. (7) takes the following form:

$$\Delta = \left(\frac{Q}{3} - \frac{P^2}{9}\right)^3 + \left(\frac{PQ}{6} - \frac{P^3}{27} - \frac{R}{2}\right)^2 > 0 \tag{8}$$

Let us recall expression (2), in which k denotes the uniform dilatation, $k > 0$ stands for expansion, and $k < 0$ stands for compression. The three invariants of $\nabla\mathbf{u}$ (P, Q, and R) can be easily expressed in terms of the three invariants of the deviatoric part of $\nabla\mathbf{u}$, denoted as P_D, Q_D, and R_D:

$$P = P_D + 3k \tag{9a}$$

[where, by definition, $P_D = (a + b + c) = 0$]

$$Q = Q_D + 3k^2 \tag{9b}$$

$$R = R_D + kQ_D + k^3 \tag{9c}$$

The substitution of the relations (9a–9c) into Eq. (8) leads to the sought criterion for the occurrence of a pair of complex-conjugate eigenvalues in the case of compressible flows, and hence for the identification of a compressible vortex, as

$$\Delta = \left(\frac{Q_D}{3}\right)^3 + \left(\frac{R_D}{2}\right)^2 > 0 \tag{10}$$

The obtained condition (10) says that there is no compressibility effect on the Δ criterion, which is determined exclusively by the deviatoric part of $\nabla\mathbf{u}$. Thus, a direct extendability of the Δ criterion to compressible flows has been confirmed.

The Δ criterion has been further enhanced in the studies of Zhou et al. [14] and Chakraborty et al. [18] dealing with the so-called swirling-strength criterion denoted as the λ_{ci} criterion. For a similar approach based on complex eigenvalues, see Berdahl and Thompson [6]. The λ_{ci} criterion stems from the Δ criterion and therefore retains the applicability to compressible flows as expressed through condition (10).

λ_2 Criterion

The approach of Jeong and Hussain [8] is formulated on dynamic considerations: namely, on the search for a pressure minimum across the vortex. The strain-rate transport equation reads

$$\frac{DS_{ij}}{Dt} - \nu S_{ij,kk} + \Omega_{ik}\Omega_{kj} + S_{ik}S_{kj} = -\frac{1}{\rho}p_{,ij} \tag{11}$$

where the pressure Hessian $p_{,ij}$ contains information on local pressure extrema. The occurrence of a local pressure minimum in a plane across the vortex requires two positive eigenvalues of the tensor $p_{,ij}$.

By removing the unsteady irrotational straining and viscous effects from the strain-rate transport equation (11), one yields the vortex-identification criterion for incompressible fluids in terms of two negative eigenvalues of $\mathbf{S}^2 + \mathbf{\Omega}^2$. Finally, a vortex is defined as a connected fluid region with two negative eigenvalues of $\mathbf{S}^2 + \mathbf{\Omega}^2$. If these eigenvalues are ordered, $\lambda_1 \geq \lambda_2 \geq \lambda_3$, the vortex-identification criterion is equivalent to the resulting condition $\lambda_2 < 0$. As noted by Chakraborty et al. [18], this condition is still not a sufficient condition to identify the plane of a local pressure minimum (in addition, a point of vanishing pressure gradient on the plane must exist).

As shown by Cucitore et al. [12], who examined the λ_2 criterion in the case of compressible fluids, additional terms occur on the left-hand side of the strain-rate transport equation, which reads

$$\rho\frac{DS_{ij}}{Dt} - \mu S_{ij,kk} + \rho(\Omega_{ik}\Omega_{kj} + S_{ik}S_{kj}) - (\lambda + \mu)u_{k,ijk} + \frac{1}{2}\left(\rho_{,j}\frac{Du_i}{Dt} + \rho_{,i}\frac{Du_j}{Dt}\right) = -p_{,ij} \tag{12}$$

where μ and λ are the first and second viscosity coefficients, respectively. The use of $\mathbf{S}^2 + \mathbf{\Omega}^2$ as an approximation of the pressure Hessian $p_{,ij}$ for compressible fluids requires discarding other terms in addition to the unsteady irrotational straining and viscous effects originally removed from the strain-rate transport equation (11), valid for incompressible flows only. According to Eq. (12), these additional terms are related to a nonzero divergence and nonzero density gradients. Consequently, the λ_2 criterion is not extendable to compressible flows.

Conclusions

Summing up, from the most popular vortex-identification schemes (Q, Δ, λ_2, and λ_{ci}), only the Δ criterion and the associated λ_{ci} criterion are extendable to compressible flows. Both criteria fulfill a specific condition that is conveniently expressed for compressible flows in terms of the invariants of the deviatoric part of $\nabla\mathbf{u}$ by the form of Eq. (10). It should be noted that if the Q criterion is a priori redefined in terms of a deviatoric part of $\mathbf{S}$ to read $(\|\mathbf{\Omega}\|^2 - \|\mathbf{S}_D\|^2)/2$, it will also be applicable to compressible flows.

Unlike the local criteria sharing a basis in the velocity-gradient tensor $\nabla\mathbf{u}$, Haller [19] provided an objective frame-independent definition of a vortex based on Lagrangian stability considerations. His analysis is, however, limited to the 3-D incompressible flows. To identify vortices in 3-D compressible variable-density flows governed by the baroclinic term (i.e., the normalized cross product of a density gradient and pressure gradient) in the vorticity equation, Zhang and Choudhury [20] proposed a new identification scheme: eigen helicity density. Further, the recent method of Kolář [21] introduces a qualitative "comeback" of vorticity to vortex identification. This approach aims at the extraction of a pure shearing motion through the decomposition

$$\nabla\mathbf{u} = \mathbf{S} + \mathbf{\Omega} = (\text{residual tensor} = \mathbf{S}_{\text{RES}} + \mathbf{\Omega}_{\text{RES}}) + (\text{shear tensor})$$

by maximizing the shear-indicating quantity

$$|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|$$

which is not affected by a nonzero uniform dilatation in the case of compressible flows. A specific portion of vorticity labeled *residual* vorticity, which is obtained after the extraction of a pure shearing motion, is proposed to represent a local intensity of the swirling motion of a vortex. This kinematic measure is free of compressibility and variable-density effects. All the line-type vortex-identification methods (Levy et al. [5], Banks and Singer [7], Sujudi and Haimes [9], Kida and Miura [10], Roth and Peikert [11], Strawn et al. [13], Peikert and Roth [15], and Roth [16]) provide a vortex skeleton in terms of vortex-core lines and are mostly free of direct compressibility effect. In general, the effect of compressibility plays an important role in many interesting problems, including the bifurcation, stability, and breakdown of compressible swirling flows (Rusak and Lee [23] and Rusak et al. [24]). This fact underlines the need for vortex-identification schemes that are applicable to compressible flows.

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